Solution to Homework 5

Total 20 points

Problem 1. (5 points) Compute the projection $\mathcal{P}_{\mathcal{C}}(\mathbf{x})$ for the following sets:

(a) (2 points) halfspace: $\mathcal{C} = \{\mathbf{x} | \mathbf{a}^\top \mathbf{x} \leq b\} (\mathbf{a} \neq 0)$;

(b) (3 points) unit ℓ_1 ball: $\mathcal{C} = \{\mathbf{x} | \|\mathbf{x}\|_1 \leq 1\}$.

Solution.

(a) If $\mathbf{a}^\top \mathbf{x} \leq b$, then $\mathcal{P}_{\mathcal{C}}(\mathbf{x}) = \mathbf{x}$. Otherwise, $\mathcal{P}_{\mathcal{C}}(\mathbf{x}) = \mathbf{x} - \frac{(\mathbf{a}^\top \mathbf{x} - \mathbf{b})\mathbf{a}}{\|\mathbf{a}\|_2^2}$.

(b) If $\|\mathbf{x}\|_1 \leq 1$ then $\mathcal{P}_{\mathcal{C}}(\mathbf{x}) = \mathbf{x}$. Otherwise, $\mathcal{C}(\mathbf{x})\mathcal{P}_{\mathcal{C}}(\mathbf{x})_i = sign(x_i)(|x_i| - \lambda)_+$ where $(\cdot)_+ = \max\{\cdot, 0\}$ and λ is the root of $\sum_{i=1}^n (|x_i| - \lambda)_+ = 1$.

Problem 2. (5 points) Suppose f is a convex and differentiable function, $\mathcal{C}$ is a closed convex set. Show that

$$\mathbf{x}^* \in \arg\min_{\mathbf{x} \in \mathcal{C}} f(\mathbf{x}) \iff \langle -\nabla f(\mathbf{x}^*), \mathbf{z} - \mathbf{x}^* \rangle \leq 0, \ \forall \ \mathbf{z} \in \mathcal{C}.$$

Solution.

$\Rightarrow$: We prove by contradiction. Suppose there exists $\mathbf{y} \in \mathcal{C}$ such that $\langle \nabla f(\mathbf{x}^*), \mathbf{y} - \mathbf{x}^* \rangle < 0$. Consider $\mathbf{z}(t) = t\mathbf{y} + (1-t)\mathbf{x}^*$ where $t \in [0, 1]$ is a parameter. Since $\mathbf{x}^*, \mathbf{y} \in \mathcal{C}$, we know $\mathbf{z}(t) \in \mathcal{C}$. Since

$$\left. \frac{\mathrm{d}}{\mathrm{d}\, t} f(\mathbf{z}(t)) \right|_{t=0} = \langle \nabla f(\mathbf{x}^* + 0 \cdot (\mathbf{y} - \mathbf{x}^*)), \mathbf{y} - \mathbf{x}^* \rangle = \langle \nabla f(\mathbf{x}^*), \mathbf{y} - \mathbf{x}^* \rangle < 0,$$

we can get for small positive t, $f(\mathbf{z}(t)) < f(\mathbf{x}^*)$.

$\Leftarrow$: Since $f(\mathbf{x})$ is convex, we have

$$f(\mathbf{x}^*) \leq f(\mathbf{z}) + \langle -\nabla f(\mathbf{x}^*), \mathbf{z} - \mathbf{x}^* \rangle \leq f(\mathbf{z}), \ \forall \ \mathbf{z} \in \mathcal{C}.$$

Problem 3. (5 points) Consider the projected gradient descent algorithm introduced in the class. Suppose that for some iteration t, $\mathbf{x}_{t+1} = \mathbf{x}_t$. Prove that in this case, $\mathbf{x}_t$ is a minimizer of the convex objective function f over the closed and convex set $\mathcal{C}$.

Solution. Let $\mathbf{y}_{t+1} = \mathbf{x}_t - \eta_t \nabla f(\mathbf{x}_t)$, then we know $\mathbf{x}_{t+1} = \mathcal{P}_{\mathcal{C}}(\mathbf{y}_{t+1})$. According to property 1 in page 8 of the slides, we have

$$\langle \mathbf{x} - \mathbf{x}_{t+1}, \mathbf{y}_{t+1} - \mathbf{x}_{t+1} \rangle \leq 0, \ \forall \ \mathbf{x} \in \mathcal{C}.$$

Since $\mathbf{x}_{t+1} = \mathbf{x}_t$ and $\mathbf{y}_{t+1} - \mathbf{x}_t = -\eta_t f(\mathbf{x}_t)$, we have

$$\langle \mathbf{x} - \mathbf{x}_t, -\eta_t f(\mathbf{x}_t) \rangle \leq 0, \ \forall \ \mathbf{x} \in \mathcal{C}.$$

According to the problem 1, we know $\mathbf{x}_t \in \arg\min_{\mathbf{x} \in \mathcal{C}} f(\mathbf{x})$.

Problem 4. (5 points) Let $\mathcal{C} \in \mathbb{R}^d$ be a nonempty closed and convex set, and let f be a strongly convex function over $\mathcal{C}$. Prove that f has a unique minimizer $\mathbf{x}^*$ over $\mathcal{C}$.

Solution. Suppose there are two minimizer $\mathbf{x}^*$ and $\mathbf{y}^*$. Since f is strongly convex, we have

$$f(\mathbf{y}^*) \geq f(\mathbf{x}^*) + \langle \nabla f(\mathbf{x}^*), \mathbf{y}^* - \mathbf{x}^* \rangle + \frac{\mu}{2} \left\| \mathbf{x}^* - \mathbf{y}^* \right\|_2^2.$$

Since $\mathbf{x}^*$ is a minimizer, according to the problem 1, we know $\langle -\nabla f(\mathbf{x}^*), \mathbf{y}^* - \mathbf{x}^* \rangle \leq 0$. Thus

$$\frac{\mu}{2} \left\| \mathbf{x}^* - \mathbf{y}^* \right\|_2^2 \leq \langle -\nabla f(\mathbf{x}^*), \mathbf{y}^* - \mathbf{x}^* \rangle + f(\mathbf{y}^*) - f(\mathbf{x}^*) \leq 0.$$

which means $\mathbf{x}^* = \mathbf{y}^*$.