Solution to Homework 4

Problem 1. (5 points) If the convex function f satisfies

$$\langle \nabla f(\mathbf{x}), \mathbf{x} - \mathbf{x}^* \rangle \geq \frac{\mu}{2} \|\mathbf{x} - \mathbf{x}^*\|_2^2 + \frac{1}{2L} \|\nabla f(\mathbf{x})\|_2^2, \quad \forall \mathbf{x}$$

where $\mathbf{x}^*$ is the minimizer, show that gradient descent with $\eta_t = \eta = \frac{1}{L}$ outputs

$$\|\mathbf{x}_t - \mathbf{x}^*\|_2^2 \leq \left(1 - \frac{\mu}{L}\right)^t \|\mathbf{x}_0 - \mathbf{x}^*\|_2^2 .$$

Solution.

$$\begin{aligned}
\|\mathbf{x}_{t+1} - \mathbf{x}^*\|_2^2 &= \left\| \mathbf{x}_t - \mathbf{x}^* - \frac{1}{L} \nabla f(\mathbf{x}_t) \right\|_2^2 \\
&= \|\mathbf{x}_t - \mathbf{x}^*\|_2^2 + \frac{1}{L^2} \|\nabla f(\mathbf{x}_t)\|_2^2 - \frac{2}{L} \langle \mathbf{x}_t - \mathbf{x}^*, \nabla f(\mathbf{x}_t) \rangle \\
&\leq \|\mathbf{x}_t - \mathbf{x}^*\|_2^2 - \frac{\mu}{L} \|\mathbf{x}_t - \mathbf{x}^*\|_2^2 \\
&= \left(1 - \frac{\mu}{L}\right) \|\mathbf{x}_t - \mathbf{x}^*\|_2^2
\end{aligned}$$

where the last inequality comes from the condition mentioned in the problem. Thus we have

$$\|\mathbf{x}_t - \mathbf{x}^*\|_2^2 \leq \left(1 - \frac{\mu}{L}\right)^t \|\mathbf{x}_0 - \mathbf{x}^*\|_2^2 .$$