Homework 7

Total 30 points

Due december 10th at 11:59pm

Problem 1. (12 points) For each of the following convex functions, compute the proximal operator:

(a) $f(\mathbf{x}) = \|\mathbf{x}\|_2$;

(b) $f(\mathbf{x}) = -\sum_{i=1}^{d} \log(x_i)$, $\mathbf{x} \in \mathbb{R}^d$;

(c) $f(\mathbf{x}) = \inf_{y \in \mathcal{C}} \|\mathbf{x} - \mathbf{y}\|_2$, where $\mathcal{C}$ is a closed convex set.

(d) $f(\mathbf{x}) = \frac{1}{2}(\inf_{y \in \mathcal{C}} \|\mathbf{x} - \mathbf{y}\|_2)^2$, where $\mathcal{C}$ is a closed convex set.

Problem 2. (5 points) (**Extended Moreau decomposition**) Suppose f is closed and convex, and $\lambda > 0$, show that

$$\mathbf{x} = \operatorname{prox}_{\lambda f}(\mathbf{x}) + \lambda \operatorname{prox}_{\frac{1}{\lambda} f^*}(\mathbf{x}/\lambda).$$

Problem 3. (13 points) Given a data matrix $\mathbf{A} \in \mathbb{R}^{n \times d}$ and data vector $\mathbf{b} \in \mathbb{R}^n$, the Lasso regression aims to solve the following the regularized least-squares problem:

$$\min_{\mathbf{x} \in \mathbb{R}^d} \frac{1}{2n} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2 + \lambda \|\mathbf{x}\|_1.$$

The homework ZIP file contains two text files, labeled `A_Lasso.txt` and `b_Lasso.txt`, that contains an $n \times d$ matrix $\mathbf{A}$ and an n-dimensional vector $\mathbf{b}$, with $n = 1000$, $d = 500$. The file `xopt.txt` contains the optimal solution to the Lasso problem with regularization parameter $\lambda = 2\sigma\sqrt{\frac{\log d}{n}}$ with $\sigma = 0.2$.

(a) (3 points) Implement the subgradient descent algorithm with both constant ($\eta = 0.2$) and decaying stepsizes ($\eta_t = 1/\sqrt{t+1}$).

(b) (3 points) Now implement the proximal gradient algorithm with constant step size $\eta = 0.2$.

(c) (4 points) For both the proximal gradient algorithm and subgradient algorithm, plot (on the same axes) the logarithm of Euclidean norm errors $\log \|\mathbf{x}_t - \mathbf{x}^*\|_2$ versus the iteration number, where $\mathbf{x}^*$ is the optimal solution from `xopt.txt`.

(d) (3 points) Comment on the convergence behavior that you see for the two methods and the relation to results established in class. Hand in your code and the report.