Homework 3

Total 20 points

Due November 5th at 11:59pm

Problem 1. (4 points) Judge whether the following functions are smooth.

(a) $f(x) = \sin x$.

(b) $f(\mathbf{x}) = \|\mathbf{x}\|_1$, $\mathbf{x} \in \mathbb{R}^d$.

Problem 2. (4 points) Judge whether the following functions are strongly convex.

(a) $f(\mathbf{x}) = \sum_{i=1}^{m} (\mathbf{a}_i^\top \mathbf{x} - b_i)^2$, $\mathbf{a}_i, \mathbf{x} \in \mathbb{R}^d$, $m > d$.

(b) $f(x_1, x_2) = 1/(x_1 x_2)$, $x_1 > 0, x_2 > 0$.

Problem 3. (12 points)

(a) Suppose that $f : \mathbb{R}^d \to \mathbb{R}$ is α-strongly convex and β-smooth for some $\beta > \alpha$. Show that $h(\mathbf{x}) = f(\mathbf{x}) - \frac{\alpha}{2}\|\mathbf{x}\|^2$ is $(\beta - \alpha)$-smooth.

(b) Suppose that $f : \mathbb{R}^d \to \mathbb{R}$ is α-strongly convex and $g : \mathbb{R}^d \to \mathbb{R}$ is β-smooth. Prove that the function $h(x) = f(x) - g(x)$ is convex if $\alpha \geq \beta$. Is the converse true?

(c) Suppose that $f : \mathbb{R}^d \to \mathbb{R}$ is μ-strongly convex and L-smooth. Show that

$$\langle \nabla f(\mathbf{x}) - \nabla f(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle \geq \frac{\mu L}{\mu + L}\|\mathbf{x} - \mathbf{y}\|_2^2 + \frac{1}{\mu + L}\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\|_2^2.$$

(hint: by the conclusion of (a), $h(\mathbf{x}) = f(\mathbf{x}) - \frac{\mu}{2}\|\mathbf{x}\|^2$ is $(L - \mu)$-smooth and convex.)