Homework 1

Total 30 points

Due October 15th at 11:59pm

Problem 1. (8 points) For each of the following sequence, determine the rate of convergence and the rate constant:

(a) $x_k = 1 + 5 \times 10^{-2k}$.

(b) $x_k = 2^{-2^k}$.

(c) $x_k = 3^{-k^2}$.

(d) $x_{k+1} = x_k/2 + 2/x_k$, $x_1 = 4$.

Problem 2. (10 points) Judge the properties of the following sets (openness, closeness, boundedness, compactness) and give their interiors, closures, and boundaries:

(a) $\mathcal{C}_1 = \emptyset$.

(b) $\mathcal{C}_2 = \mathbb{R}^n$.

(c) $\mathcal{C}_3 = \{(x, y)^\top | x \geq 0, y > 0\}$.

(d) $\mathcal{C}_4 = \{k | k \in \mathbb{Z}\}$.

(e) $\mathcal{C}_5 = \{(1/k, \sin k) | k \in \mathbb{Z}\}$.

Problem 3. (4 points) Compute the **gradient** and the **Hessian** of the following functions (write in vector or matrix form, rather than entrywise), give details:

(a) $f(\mathbf{x}) = (\mathbf{a}^\top \mathbf{x})(\mathbf{b}^\top \mathbf{x})$.

(b) $f(\mathbf{x}) = \frac{1}{2}||\mathbf{A}\mathbf{x} - \mathbf{b}||_2^2$.

Problem 4. (8 points) Which of the following sets are convex? Explain your answer.

(a) A wedge, i.e., a set of the form $\{\mathbf{x} \in \mathbb{R}^n | \mathbf{a}_1^\mathrm{T} \mathbf{x} \leq b_1, \mathbf{a}_2^\mathrm{T} \mathbf{x} \leq b_2\}$.

(b) The set of points closer to a given point than a given set, i.e., $\{\mathbf{x} | \|\mathbf{x} - \mathbf{x}_0\|_2 \leq \|\mathbf{x} - \mathbf{y}\|_2 \text{ for all } \mathbf{y} \in S\}$ where $S \subseteq \mathbb{R}^n$.

(c) The set of points closer to one set than another, i.e., $\{\mathbf{x} | \mathbf{dist}(\mathbf{x}, S) \leq \mathbf{dist}(\mathbf{x}, T)\}$ where $S, T \subseteq \mathbb{R}^n$, and $\mathbf{dist}(\mathbf{x}, S) = \inf\{\|\mathbf{x} - \mathbf{z}\|_2 | \mathbf{z} \in S\}$.

(d) The set of points whose distance to $\mathbf{a}$ does not exceed a fixed fraction θ of the distance to $\mathbf{b}$, i.e., the set $\{\mathbf{x} | \|\mathbf{x} - \mathbf{a}\|_2 \leq \theta \|\mathbf{x} - \mathbf{b}\|_2\}$ ($\mathbf{a} \neq \mathbf{b}$ and $0 \leq \theta \leq 1$).