Notes for Lecture 9

Scribe: Tingkai Jia

1 Convergence of Proximal Gradient Descent

Lemma 1. *Let* $\mathbf{y}^+ = \mathrm{prox}_{\frac{1}{L}h}(y - \frac{1}{L}\nabla f(\mathbf{y}))$*, then*

$$F(\mathbf{y}^+) - F(\mathbf{x}) \le \frac{L}{2}\|\mathbf{x}-\mathbf{y}\|_2^2 - \frac{L}{2}\|\mathbf{x}-\mathbf{y}^+\|_2^2 - g(\mathbf{x},\mathbf{y})$$

where $g(\mathbf{x},\mathbf{y}) = f(\mathbf{x}) - f(\mathbf{y}) - \langle \nabla f(\mathbf{y}), \mathbf{x}-\mathbf{y}\rangle$.

Proof. Define $\phi(\mathbf{z}) = f(\mathbf{y}) + \langle \nabla f(\mathbf{y}), \mathbf{z}-\mathbf{y}\rangle + \frac{L}{2}\|\mathbf{z}-\mathbf{y}\|_2^2 + h(\mathbf{z})$. It is true that $\mathbf{y}^+ = \arg\min_{\mathbf{z}} \phi(\mathbf{z})$ and $\phi(\mathbf{z})$ is L-strongly convex, which means

$$\phi(\mathbf{x}) \ge \phi(\mathbf{y}^+) + \frac{L}{2}\|\mathbf{x}-\mathbf{y}^+\|_2^2.$$

From the smoothness of f, we have

$$\begin{aligned}\phi(\mathbf{y}^+) &= f(\mathbf{y}) + \langle \nabla f(\mathbf{y}), \mathbf{y}^+-\mathbf{y}\rangle + \frac{L}{2}\|\mathbf{y}^+-\mathbf{y}\|_2^2 + h(\mathbf{y}^+)\\ &\ge f(\mathbf{y}^+) + h(\mathbf{y}^+) = F(\mathbf{y}^+).\end{aligned}$$

Now we get

$$\phi(\mathbf{x}) \ge F(\mathbf{y}^+) + \frac{L}{2}\|\mathbf{x}-\mathbf{y}^+\|_2^2$$

which together with the definition of $\phi(\mathbf{x})$ gives

$$\begin{aligned}f(\mathbf{y}) + \langle \nabla f(\mathbf{y}), \mathbf{x}-\mathbf{y}\rangle + h(\mathbf{x}) + \frac{L}{2}\|\mathbf{x}-\mathbf{y}\|_2^2 &\ge F(\mathbf{y}^+) + \frac{L}{2}\|\mathbf{x}-\mathbf{y}^+\|_2^2\\ f(\mathbf{x}) + h(\mathbf{x}) - g(\mathbf{x},\mathbf{y}) + \frac{L}{2}\|\mathbf{x}-\mathbf{y}\|_2^2 &\ge F(\mathbf{y}^+) + \frac{L}{2}\|\mathbf{x}-\mathbf{y}^+\|_2^2\\ F(\mathbf{x}) - g(\mathbf{x},\mathbf{y}) + \frac{L}{2}\|\mathbf{x}-\mathbf{y}\|_2^2 &\ge F(\mathbf{y}^+) + \frac{L}{2}\|\mathbf{x}-\mathbf{y}^+\|_2^2.\end{aligned}$$

□

Theorem 1 (PGD for convex problems)**.** *Suppose* f *is convex and* L*-smooth. If* $\eta_t \equiv 1/L$*, then*

$$F(\mathbf{x}_t) - F(\mathbf{x}_*) \le \frac{L\|\mathbf{x}_0 - \mathbf{x}_*\|_2^2}{2t}.$$

Proof. With Lemma 1 in mind, set $\mathbf{x} = \mathbf{x}_*$, $\mathbf{y} = \mathbf{x}_t$ to obtain

$$\begin{aligned}F(\mathbf{x}_{t+1}) - F(\mathbf{x}_*) &\le \frac{L}{2}\|\mathbf{x}_t-\mathbf{x}_*\|_2^2 - \frac{L}{2}\|\mathbf{x}_{t+1}-\mathbf{x}_*\|_2^2 - g(\mathbf{x}_*,\mathbf{x}_t)\\ &\le \frac{L}{2}\|\mathbf{x}_t-\mathbf{x}_*\|_2^2 - \frac{L}{2}\|\mathbf{x}_{t+1}-\mathbf{x}_*\|_2^2.\end{aligned}$$

Apply it recursively and add up all inequalities to get

$$\sum_{k=0}^{t-1} \left(F(\mathbf{x}_{k+1}) - F(\mathbf{x}_*)\right) \leq \frac{L}{2}\|\mathbf{x}_0 - \mathbf{x}_*\|_2^2 - \frac{L}{2}\|\mathbf{x}_t - \mathbf{x}_*\|_2^2,$$

with $t\left(F(\mathbf{x}_t) - F(\mathbf{x}_*)\right) \leq \sum_{k=0}^{t-1}\left(F(\mathbf{x}_{k+1}) - F(\mathbf{x}_*)\right)$, we finally get

$$F(\mathbf{x}_t) - F(\mathbf{x}_*) \leq \frac{L\|\mathbf{x}_0 - \mathbf{x}_*\|_2^2}{2t}.$$

□

Theorem 2 (PGD for strongly convex problems)**.** *Suppose f is μ-strongly convex and L-smooth. if $\eta_t \equiv 1/L$, then*

$$\|\mathbf{x}_t - \mathbf{x}_*\|_2^2 \leq \left(1 - \frac{\mu}{L}\right)^t \|\mathbf{x}_0 - \mathbf{x}_*\|_2^2.$$

Proof. With Lemma 1 in mind, set $\mathbf{x} = \mathbf{x}_*$, $\mathbf{y} = \mathbf{x}_t$ to obtain

$$\begin{aligned} F(\mathbf{x}_{t+1}) - F(\mathbf{x}_*) &\leq \frac{L}{2}\|\mathbf{x}_t - \mathbf{x}_*\|_2^2 - \frac{L}{2}\|\mathbf{x}_{t+1} - \mathbf{x}_*\|_2^2 - g(\mathbf{x}_*, \mathbf{x}_t) \\ &\leq \frac{L-\mu}{2}\|\mathbf{x}_t - \mathbf{x}_*\|_2^2 - \frac{L}{2}\|\mathbf{x}_{t+1} - \mathbf{x}_*\|_2^2. \end{aligned}$$

This taken collectively with $F(\mathbf{x}_{t+1}) - F(\mathbf{x}_*) \geq 0$ yields

$$\|\mathbf{x}_{t+1} - \mathbf{x}_*\|_2^2 \leq \left(1 - \frac{\mu}{L}\right)\|\mathbf{x}_t - \mathbf{x}_*\|_2^2.$$

Applying it recursively concludes the proof. □