Homework 8

Deadline: December 11th

Problem 1. Let $F(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} f_i(\mathbf{x})$, where $f_i(\mathbf{x})$ is differentiable and L-smooth. Suppose j is uniformly sampled from $\{1, 2, \ldots, n\}$. Show that

$$\mathbb{E}[\|\nabla f_j(\mathbf{x})\|_2^2] \leq L^2 \mathbb{E}[\|\mathbf{x} - \mathbf{x}^*\|_2^2] + \mathbb{E}[\|\nabla f_j(\mathbf{x}) - \nabla F(\mathbf{x}))\|_2^2]$$

where $\mathbf{x}^*$ is a minimizer of $F(\mathbf{x})$.

Problem 2. In this problem, we study a stochastic gradient method with a projection step. Let $f : \mathbb{R}^d \to \mathbb{R}$ be differentiable and μ-strongly convex, and let $\mathcal{C}$ be a closed, convex set. Consider the projected stochastic gradient method

$$\mathbf{x}_{t+1} = \mathcal{P}_{\mathcal{C}}(\mathbf{x}_t - \eta_t G(\mathbf{x}_t)),$$

where $G(\mathbf{x}_t)$ is an unbiased estimate of $\nabla f(\mathbf{x}_t)$. Assume that the randomness in $G(\mathbf{x}_t)$ is independent of all past randomness in the algorithm. Letting $\mathbf{x}^* = \arg\min_{\mathbf{x} \in \mathcal{C}} f(\mathbf{x})$, prove that the iterates satisfy the bound

$$\mathbb{E}[\|\mathbf{x}_{t+1} - \mathbf{x}^*\|_2^2] \leq (1 - 2\eta_t \mu)\mathbb{E}[\|\mathbf{x}_t - \mathbf{x}^*\|_2^2] + \eta_t^2 B^2$$

where $B^2 = \sup_{\mathbf{x} \in \mathcal{C}} \mathbb{E}\,\|G(\mathbf{x})\|_2^2$.

Problem 3. Prove the conclusion on page 10 of the slides.