Notes for Lecture 2

Theorem. A function $f(\mathbf{x})$ is convex if and only if its epigraph is a convex set.
Proof. *Part I*: Suppose f is convex. For any $(\mathbf{x}_1, u_1)$ and $(\mathbf{x}_2, u_2)$ in *epi* f, $\alpha \in [0,1]$, we have

$$f(\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2) \leq \alpha f(\mathbf{x}_1) + (1-\alpha) f(\mathbf{x}_2) \leq \alpha u_1 + (1-\alpha) u_2,$$

which means *epi* f is a convex set.
Part II: Suppose *epi* f is convex. Notice that for any $\mathbf{x}_1, \mathbf{x}_2 \in$ *dom* f, we have $(\mathbf{x}_1, f(\mathbf{x}_1))$ and $(\mathbf{x}_2, f(\mathbf{x}_2))$ belong to *epi* f. Then for any $\alpha \in [0,1]$, we can get

$$(\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2, \alpha f(\mathbf{x}_1) + (1-\alpha) f(\mathbf{x}_2)) \in \textit{epi } f,$$

which means $f(\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2) \leq \alpha f(\mathbf{x}_1) + (1-\alpha) f(\mathbf{x}_2))$.

Theorem. If $f(\mathbf{x}, \mathbf{y})$ is convex in $(\mathbf{x}, \mathbf{y})$ and C is a convex set, then

$$g(\mathbf{x}) = \inf_{\mathbf{y} \in C} f(\mathbf{x}, \mathbf{y})$$

is convex.
Proof. Suppose $\mathbf{x}_1, \mathbf{x}_2 \in \text{dom } g$. For any $\epsilon > 0$, there are $\mathbf{y}_1, \mathbf{y}_2 \in C$, such that

$$\begin{aligned} f(\mathbf{x}_1, \mathbf{y}_1) \leq & g(\mathbf{x}_1) + \epsilon; \\ f(\mathbf{x}_2, \mathbf{y}_2) \leq & g(\mathbf{x}_2) + \epsilon; \end{aligned}$$

Since $f(\mathbf{x}, \mathbf{y})$ is convex, we have

$$f(\theta\mathbf{x}_1 + (1-\theta)\mathbf{x}_2, \theta\mathbf{y}_1 + (1-\theta)y_2) \leq \theta f(\mathbf{x}_1, y_1) + (1-\theta) f(\mathbf{x}_2, y_2).$$

Thus,

$$\begin{aligned} g(\theta\mathbf{x}_1 + (1-\theta)\mathbf{x}_2) = & \inf_{\mathbf{y} \in C} f(\theta\mathbf{x}_1 + (1-\theta)x_2, \mathbf{y}) \\ \leq & f(\theta\mathbf{x}_1 + (1-\theta)\mathbf{x}_2, \theta\mathbf{y}_1 + (1-\theta)\mathbf{y}_2) \\ \leq & \theta f(\mathbf{x}_1, \mathbf{y}_1) + (1-\theta) f(\mathbf{x}_2, \mathbf{y}_2) \\ \leq & \theta g(\mathbf{x}_1) + (1-\theta) g(\mathbf{x}_2) + \epsilon. \end{aligned}$$

Since this holds for any $\epsilon > 0$, we have

$$g(\theta\mathbf{x}_1 + (1-\theta)\mathbf{x}_2) \leq \theta g(\mathbf{x}_1) + (1-\theta) g(\mathbf{x}_2).$$